Published in the Proceedings of the National Academy of Sciences. Vol. 40, No. 2 pp. 128-132 February, 1954.

DIFFRACTION BY A STRIP

by

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Technical Report No. 1

CALIFORNIA INSTITUTE OF TECHNOLOGY

Pasadena, California.

A Technical Report to the Office of Naval Research

Prepared for

Office of Naval Research Contract

Nonr 220(14)

NR 071-262

November 1953

On Diffraction by a Strip

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Introduction

The problem of diffraction by an infinite strip or slit has been the subject of several investigations[1]. There are at least two "exact" methods for attacking this problem. One of these is the integral equation method,[2] the other the Fourier-Lamé method.[3] The integral equation obtained for this problem cannot be solved in closed form; expansion of the solution in powers of the ratio (strip width/wavelength) leads to useful formulas for low frequencies. In the Fourier-Lamé method the wave equation is separated in coordinates of the elliptic cylinder, the solution appears as an infinite series of Mathieu functions, and the usefulness of the result is limited by the convergence of these infinite series, and by the available tabulation of Mathieu functions.

The variational technique developed by Levine and Schwinger avoids some of the difficulties of the above-mentioned methods and, at least in principle, is capable of furnishing good approximations for all frequency-ranges. The scattered field may be represented as the effect of the current induced in the strip, and it has been proved by Levine and Schwinger[4] that it is possible to represent the amplitude of the far-zone scattered field in terms of the induced current in a form which is stationary with respect to small variations of the current about the true current. Substitution, in this representation, of a rough approximation for the current may give a remarkably good approximation of the far-zone scattered field amplitude. In this note we assume a normally incident field polarized parallel to the generators of the strip. As a rough approximation, we take a uniform density of the current induced in the strip. Since the incident magnetic field is constant over the strip, Fock's theory[5] may be cited in support of the uniformity of the current distribution, except near the edges where the behaviour of the field[6] indicates an infinite current density. A more detailed analysis of the current, by Moullin and Phillips[3], is available but was not used here.

Once the (approximate) amplitude of the far-zone field has been obtained, the scattering cross-section may be found by the application of the scattering theorem[4,7,8] which relates this cross-section to the imaginary part of the amplitude of the far-zone scattered field along the central line of the umbral region. In spite of the crude approximation adopted for the induced current, the scattering cross-section shows a fair agreement with other available results.

Integral Equation

We assume a plane wave with harmonic time dependence $\exp(-i\omega t)$, normally incident on the perfectly conducting infinitesimally thin strip of width $2a$,

$$z = 0, \quad -a \leq y \leq a \; . \tag{1}$$

We further assume that the incident wave is polarized parallel to the edges of the strip (i.e. to the x axis) so that the only nonvanishing components of its electric field $\underset{\sim}{E}$, magnetic field $\underset{\sim}{H}$, and complex Poynting vector $\underset{\sim}{S}$ (all measured in MKS units, bars denoting conjugate complex numbers) are

$$E_x^{\,i} = \exp(ikz) \tag{2}$$

$$H_y^{\,i} = k/(\omega\mu) \exp(ikz) \tag{3}$$

$$S_z^{\,i} = E_x^{\,i}\, \overline{H_y^{\,i}} = k/(\omega\mu) \tag{4}$$

The scattered electric field is again parallel to the x axis, and the total electric field is the sum of the incident and scattered field,

$$E_x(y,z) = E_x^{\,i}(y,z) + E_x^{\,sc}(y,z) \tag{5}$$

The boundary condition on the screen is the vanishing of the electric field

$$E_x(y,o) = 0 \quad \text{for} \quad -a<y<a \; ; \tag{6}$$

and the scattered field, E_x^{sc}, must represent, at large distances from the screen, an outgoing cylindrical wave (Sommerfeld's radiation condition).

The expression of the total electric field in terms of the induced current $K_x(y)$ is

$$E_x(y,z) = \exp(ikz) - \omega\mu/4 \int_{-a}^{a} H_o^{(1)} \left(k \left[(y - y')^2 + z^2\right]^{1/2}\right) K_x(y')dy' \tag{7}$$

$i/4 \; H_o^{(1)}(k \left[(y-y')^2 + z^2\right]^{1/2})$ being the two-dimensional free-space Green's function. The scattered field in (7) certainly satisfies Sommerfeld's radiation condition; in order that it also satisfies (6), we must have

$$1 = \omega\mu/4 \int_{-a}^{a} H_o^{(1)} (k \, |y-y'| \,) \, K_x(y') \, dy' \qquad |y| \leq a \tag{8}$$

and this is the integral equation of our problem.

Far-Zone Scattered Field Amplitude

We define the far-zone scattered field amplitude $A(\phi)$ by

$$E_x^{sc} \sim i/4 \left[2/(\pi i k \rho)\right]^{1/2} \exp(ik\rho) \, A(\phi) \qquad k\rho \longrightarrow \infty \tag{9}$$

where $\rho \cos \phi = z$ and $\rho \sin \phi = -y$; this expression represents a cylindrical outgoing wave of "amplitude" A.

Since

$$H_o^{(1)} \left(k\left[(y-y')^2 + z^2\right]^{1/2}\right) \left[2/(\pi i k \rho)\right]^{1/2} \exp(ik\rho + iky' \sin \phi) \tag{10}$$

when $k\rho \longrightarrow \infty$, we have from (7), for large $k\rho$,

$$E_x^{sc} \sim i/4 \left[2/(\pi i k \rho)\right]^{1/2} \exp(ik\rho) \int_{-a}^{a} i \omega \mu \exp(iky' \sin \phi) \, K_x(y') \, dy' \tag{11}$$

Comparing (9) and (11) we get

$$A(\phi) = i\omega\mu \int_{-a}^{a} \exp(iky' \sin \phi)\, K_x(y')\, dy' \tag{12}$$

A Scattering Theorem

If P denotes the scattered energy flux per unit length, then the scattering cross-section σ is defined by

$$\sigma = P/S \tag{13}$$

where $S = 1/2 \text{ Re } S_z^i = k/(2\omega\mu)$ is the incident energy flux per unit area.

$$P = 1/2 \text{ Re} \int_{-a}^{a} E_x^i\,(y',0)\, \overline{K}_x(y')\, dy' = 1/2 \text{ Re} \int_{-a}^{a} K_x(y')\, dy' \tag{14}$$

since $E_x^i = 1$ at $z = 0$. Hence,

$$\sigma = \omega\mu/k \text{ Re} \int_{-a}^{a} K_x(y')\, dy' \tag{15}$$

But from (12)

$$A(0) = i\omega\mu \int_{-a}^{a} K_x(y')\, dy' \tag{16}$$

Therefore, comparing (15) and (16) we get

$$\sigma = \text{Im } A(0)/k \tag{17}$$

It may be remarked that in the limit for very high frequencies, the cross-section (17) turns out to be twice the projected area. At first sight the factor two may seem inconsistent with geometrical optics, but actually the discrepancy is due to the different definitions of the scattering cross section.[9]

Variational Principle

We multiply the integral equation (8) by $K_x(y)$, integrate from $y = -a$

to $y = +a$, divide by the square of $\int_{-a}^{a} K_x(y)dy$, and then recall (16). Thus we obtain

$$A(0) = 4i \frac{\left[\int_{-a}^{a} K_x(y)\, dy\right]^2}{\int_{-a}^{a}\int_{-a}^{a} K_x(y)\, H_0^{(1)}(k\,|y-y'|)\, K_x(y')\, dy\, dy'} \tag{18}$$

In Appendix A it is shown that the expression (18) is stationary with respect to small variations of K_x about the true K_x which satisfies the integral equation (8). Consequently, if we substitute a reasonable trial function for the unknown K_x in (18), we expect to obtain a reasonably accurate value of $A(0)$.

Trial Function

According to Fock's theory[5] the current induced in the central region of the strip is equal to twice the tangential component of the incident magnetic field. And according to Bouwkamp[6] H_y must have a singularity like $(a^2-y^2)^{-1/2}$ at the edges. This certainly conforms to the detailed information Moullin and Phillips[3] reported. In the present paper we take for a trial function $K_x(y) = 1$, thus ignoring the effect of the edges.

Scattering Cross Section

With $K_x(y) = 1$, computation (see Appendix B) shows that (18) becomes

$$\left[A(0)\right]^{-1} = (4iak)^{-1}\left\{\int_0^{2ka} H_0^{(1)}(t)dt - H_1^{(1)}(2ka) + (\pi ika)^{-1}\right\} \tag{19}$$

And applying the theorem (17) to (19), we get

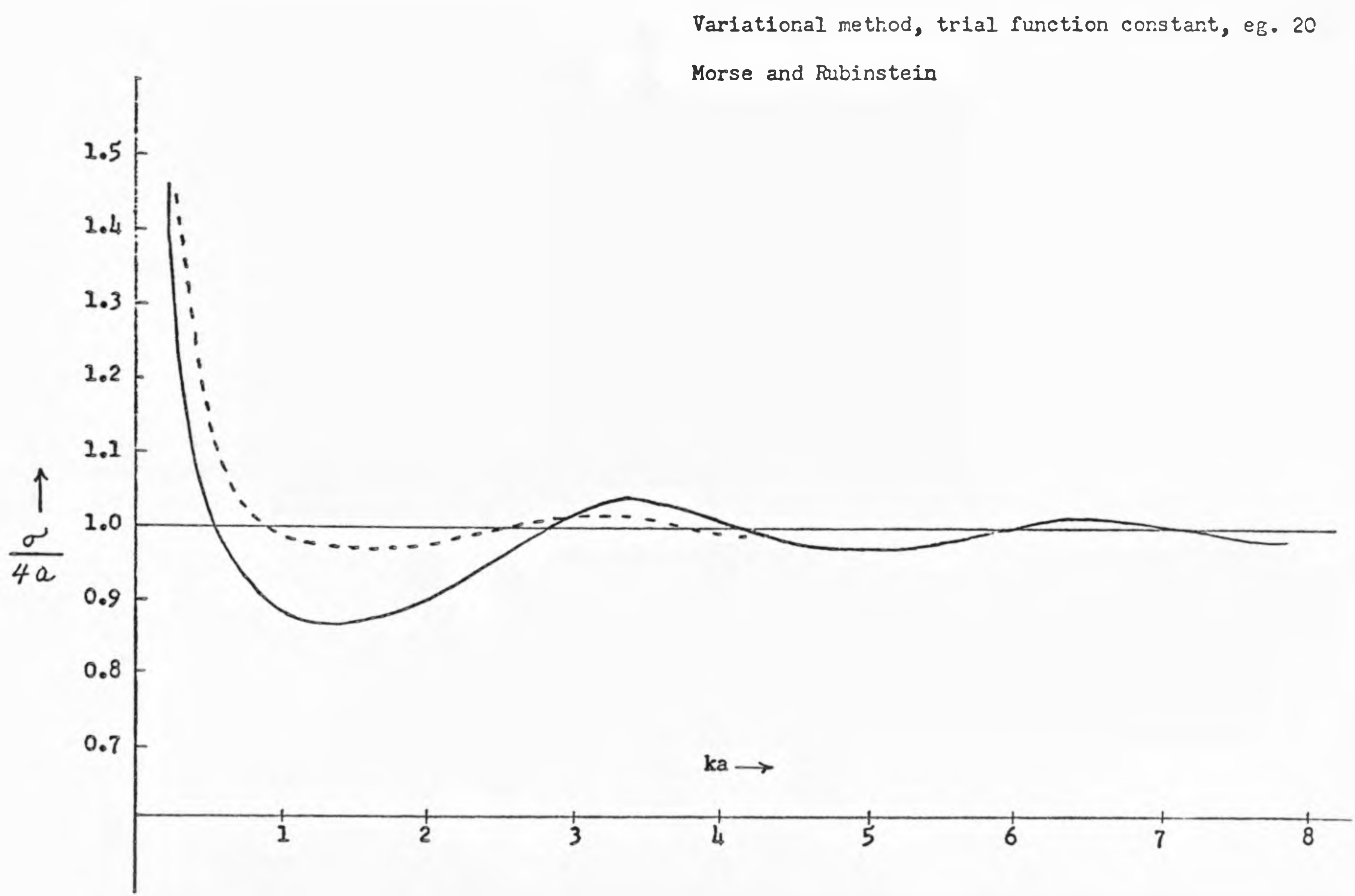
$$\sigma/(4a) = \frac{\int_0^{2ka} J_o(t)dt - J_1(2ka)}{\left[\int_0^{2ka} J_o(t)dt - J_1(2ka)\right]^2 + \left[\int_0^{2ka} Y_o(t)dt - Y_1(2ka) - (\pi ka)^{-1}\right]^2} \qquad (20)$$

The integrals appearing here have been tabulated.[10]

Results

A plot of $\sigma/(4a)$ versus ka is shown in the graph. This is a plot of eq.(20). For small ka we have Rayleigh scattering. As ka increases the curve performs a damped oscillation about $\sigma/(4a) = 1$. And for $ka \to \infty$ it can be shown by means of asymptotic representations of Bessel functions and their integrals that $\sigma \to 4a$.

According to Babinet's principle[11] the problem we have discussed is complementary to the problem of scattering by a slit for a normally incident wave polarized perpendicular to the axis of the slit. For intermediate values of ka our curve behaves qualitatively as Morse and Rubinstein's[1]; the quantitative agreement is not very good, the deviation being due to our choice of an overly simplified trial function. It is quite remarkable how such a rough approximation of the induced current yields fairly good results over the entire spectrum.



Variational method, trial function constant, eg. 20
Morse and Rubinstein
σ/4a
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
ka →
1
2
3
4
5
6
7
8

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Appendix A

We show here that $A(0)$ of (18) is stationary with respect to small variations of $K_x(y)$ about the true $K_x(y)$ which satisfies the integral equation (8).

From (18) and the calculus of variations it follows that

$$\delta A \int_{-a}^{a}\int_{-a}^{a} K_x(y)\, H_o^{(1)}(k\,|y-y'|)\, K_x(y')\, dy\, dy'$$

$$+ A \int_{-a}^{a}\int_{-a}^{a} \delta K_x(y)\, H_o^{(1)}(k\,|y-y'|)\, K_x(y')\, dy\, dy'$$

$$+ A \int_{-a}^{a}\int_{-a}^{a} K_x(y)\, H_o^{(1)}(k\,|y-y'|)\, \delta K_x(y')\, dy\, dy' =$$

$$2 \int_{-a}^{a} \delta K_x(y)\, dy \int_{-a}^{a} K_x(y)\, dy \;.$$

Rearranging terms we get

$$\delta A \int_{-a}^{a} K_x(y)\, H_o^{(1)}(k\,|y-y'|)\, K_x(y')\, dy\, dy' =$$

$$\int_{-a}^{a} \delta K_x(y)\, dy \left\{ \int_{-a}^{a} K_x(y)\, dy - A \int_{-a}^{a} H_o^{(1)}(k\,|y-y'|)\, K_x(y')\, dy' \right\}$$

$$+ \int_{-a}^{a} \delta K_x(y')\, dy' \left\{ \int_{-a}^{a} K_x(y)\, dy - A \int_{-a}^{a} H_o^{(1)}(k\,|y-y'|)\, K_x(y)\, dy \right\} .$$

The right side of this equation disappears for small arbitrary variations $\delta K_x(y)$ because the quantities in curly brackets are zero by virtue of (12), i.e., $A(0) = i\omega\mu \int_{-a}^{a} K_x(y) dy$, and the integral equation (8). Consequently $\delta A = 0$ and the expression (18) is stationary.

Appendix B

For trial function $K_x(y)$ = constant, (18) becomes

$$\frac{1}{A(0)} = \frac{1}{16\, i\, a^2} \int_{-a}^{a} \int_{-a}^{a} H_o^{(1)}\, (k|y-y'|)\, dy\, dy'$$

Put $k(y+y') = s$, $k\,(y-y') = t$. Then

$$\frac{1}{A(0)} = \frac{1}{32\, i\, (ka)^2} \iint_S H_o^{(1)}\, (\,|t|\,)\, ds\, dt$$

where S is the square with vertices $(\pm\, 2ka, 0)$, $(0, \pm\, 2ka)$.

Let Q be the quadrant $s > 0$, $t > 0$, $s + t \leq 2ka$. Then

$$\frac{1}{A(0)} = \frac{1}{8\, i\, (ka)^2} \iint_Q H_o^{(1)}\, (t)\, dt = \frac{1}{8\, i\, (ka)^2} \int_0^{2ka} (2ka - t) H_o^{(1)}(t) dt.$$

Since $\displaystyle\int_0^{2ka} t\, H_o^{(1)}(t)dt = 2ka\, H_1^{(1)},\, (2ka) + \frac{2i}{\pi}$, the above equation becomes

$$\frac{1}{A(0)} = \frac{1}{4\, i\, ka} \left\{ \int_0^{2ka} H_o^{(1)}(t)dt - H_1^{(1)}(2ka) + \frac{1}{\pi\, i\, ka} \right\} .$$

And this is (19).